metal backed) are not significantly affected when the slab thickness is a very small fraction of the radius of curvature. However, the presence of a grounded layer of dielectric on the inner side of the slab leads to a resonant enhancement in the fractional change in the propagation constant. In the region where magnetostatic approximation is approximately valid, the "curvature loss" is expected to be negligible.

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On the Resonant Frequency of a Reentrant Cylindrical Cavity

MAREK JAWORSKI

Abstract—**A new efficient method determining the resonant frequency of a reentrant cylindrical cavity is suggested. The method is based on solving the Helmholtz equation within two cavity regions and matching the solutions across the boundary surface. Contrary to similar formulations published previously, the continuity conditions on the boundary are imposed in a rigorous way. As a result, the solution is obtained in a form of successive approximations converging to the exact resonant frequency when a number of iterations tend toward infinity. Numerical examples are given for a few reentrant cavities of typical dimensions. Comparison is also made with experimental data as well as other theoretical results.**

I. INTRODUCTION

REENTRANT cylindrical cavities, widely used for a number of years, have recently found a new application in solid-state devices, particularly Gunn and tunnel diode oscillators. Simultaneously, a renewed interest in approximate methods determining the resonant frequency of such cavities has been observed. In some applications it is sufficient to consider a simple equivalent circuit, usually based on TEM coaxial line and lumped capacitance [1]–[5]. In general, however, more sophisticated methods are needed in order to evaluate the resonant frequency with reasonable accuracy [6]–[9].

Recently, a new interesting approach has been suggested by Williamson [9]. In his method, the magnetic field in both regions of the reentrant cavity is excited by the "aperture" electric field given on the interface $r=a$ (see Fig. 1). The resonant frequency is then found by matching the magnetic fields across the interface and solving the appropriate transcendental equation.

The above formulation, being in fact an improvement of Hansen's approach [6], is numerically simple and provides more accurate results than the solutions published previously. Nevertheless, the main disadvantage of both Williamson's and Hansen's method is due to the fact that the aperture field, which is generally not known, has to be included in the transcendental equation. In the paper of Williamson [9], the solution of the corresponding cylindrical antenna problem has been suggested as a suitable approximation for the electric field on the interface $r=a$. Unfortunately, such an approximation is sufficiently accurate for narrow-gap cavities only. Moreover, the solution of the antenna problem, as formulated for an unbounded region, may not be adequate for resonant systems, particularly in the cases when the outer diameter of

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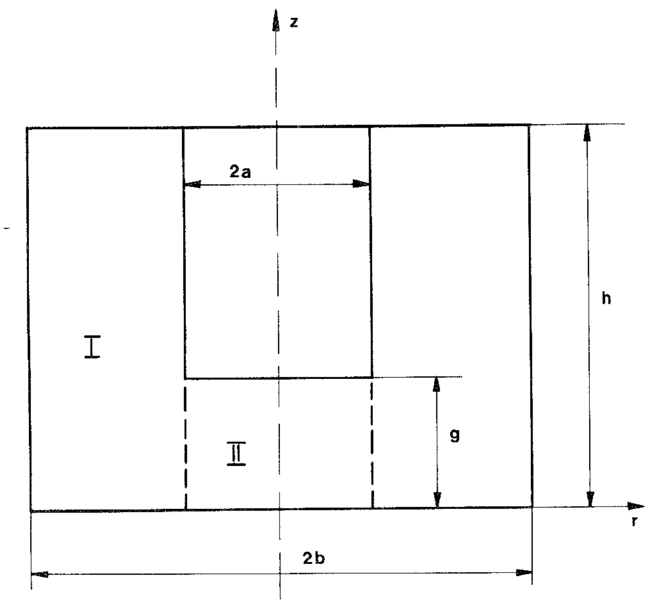

Fig. 1. Cross section of the reentrant cavity.

the cavity $2b$ is not much larger than the inner diameter $2a$. On the other hand, imposing the continuity condition for one point only [6] or for the average value of the field [9] is evidently insufficient, since the field has to be continuous in each point on the boundary surface.

In order to avoid the foregoing disadvantages, we have developed a method which is more general and simultaneously more rigorous than various formulations published until now. Since no approximation of the aperture field is required, the method outlined below is very accurate and may be applied to reentrant cavities of arbitrary dimensions. A simple numerical procedure provides a sequence of successive approximations approaching the exact solution as a limit.

II. Formulation of the Problem

The cross section of the cylindrical reentrant cavity is shown in Fig. 1. It is convenient to divide the cavity into two regular regions along the surface $r=a$. It may then be assumed that the magnetic field in each region is excited by surface magnetic currents equivalent to the electric field incident on the boundary surface (aperture) $r=a$ [10]. Since we are mainly interested in axially symmetric fundamental modes, we confine our considerations to the solutions which are independent of the angular coordinate φ.

In particular, for the magnetic field on the aperture, we obtain

$$H_\varphi^{\mathrm{I}}(a,z)=\frac{ik}{Z}\int_0^g G^{I}(a,a,z,z')E_z(a,z')\,dz', \qquad a<r<b \tag{1a}$$

and

$$H_\varphi^{\mathrm{II}}(a,z)=\frac{-ik}{Z}\int_0^g G^{II}(a,a,z,z')E_z(a,z')\,dz', \quad 0<r<a \tag{1b}$$

where

$G(r,r',z,z')$ two-dimensional Green function of the Helmholtz operator in cylindrical coordinate system,
$E_z(a,z')$ component of the electric field tangential to the surface $r=a$,
$k=2\pi/\lambda$ free-space propagation constant,
Z free-space impedance.

It can be shown that

$$G^{\mathrm{I}}(a,a,z,z')=-\frac{1}{h}\left[G_0^{\mathrm{I}}+2\sum_{m=1}^{\infty}\cos(m\pi z/h)\cdot\cos(m\pi z'/h)G_m^{\mathrm{I}}\right] \tag{2a}$$

$$G_m^{\mathrm{I}}=\begin{cases}\dfrac{J_1(va)Y_0(vb)-Y_1(va)J_0(vb)}{v[J_0(va)Y_0(vb)-Y_0(va)J_0(vb)]}, & k>m\pi/h\\[2ex] \dfrac{I_1(va)K_0(vb)+K_1(va)I_0(vb)}{v[I_0(va)K_0(vb)-K_0(va)I_0(vb)]}, & k<m\pi/h\end{cases}$$

$$v=\sqrt{|k^2-(m\pi/h)^2|}\,, \qquad m=0,1,2\cdots.$$

$J_n(x)$ and $Y_n(x)$ denote the Bessel functions of the first and second kind, respectively, while $I_n(x)$ and $K_n(x)$ denote the modified Bessel function of the first and second kind, respectively.

Similarly

$$G^{\mathrm{II}}(a,a,z,z')=\frac{1}{g}\left[G_0^{\mathrm{II}}+2\sum_{n=1}^{\infty}\cos(n\pi z/g)\cdot\cos(n\pi z'/g)G_n^{\mathrm{II}}\right] \tag{2b}$$

where

$$G_n^{\mathrm{II}}=\begin{cases}\dfrac{J_1(ua)}{uJ_0(ua)}, & k>n\pi/g\\[2ex] \dfrac{I_1(ua)}{uI_0(ua)}, & k<n\pi/g\end{cases}$$

$$u=\sqrt{|k^2-(n\pi/g)^2|}\,, \qquad n=0,1,2,\cdots.$$

Since the Green function is in fact the kernel of the resolvent operator R, it is convenient to introduce a brief operator notation

$$R\chi(a,z)=\frac{ik}{Z}\int_0^g G(a,a,z,z')\chi(a,z')\,dz' \tag{3}$$

where $\chi(a,z)$ is the arbitrary function defined for $z\in(0,g)$. Thus one can rewrite (1(a)) and (1(b)) in a simple form

$$H_\varphi^{\mathrm{I}}(a,z)=R^{\mathrm{I}}E_z(a,z) \tag{4a}$$

and

$$H_\varphi^{\mathrm{II}}(a,z)=-R^{\mathrm{II}}E_z(a,z). \tag{4b}$$

The resonance condition follows directly from the continuity condition for the magnetic field on the boundary

$$H_\varphi^{\mathrm{I}}(a,z)-H_\varphi^{\mathrm{II}}(a,z)=0 \tag{5}$$

or in a more compact form

$$F(z)=0 \tag{6}$$

where

$$F(z)=H_\varphi^{\mathrm{I}}(a,z)-H_\varphi^{\mathrm{II}}(a,z)$$
$$=R^{\mathrm{I}}E_z(a,z)+R^{\mathrm{II}}E_z(a,z). \tag{7}$$

Proper approach to the solution of (6) is the crucial point of the method, since any approximation made at this moment strongly influences the final results. The main difficulty arises from the fact that (6), from which the resonant frequency is to be determined, must be satisfied for every z belonging to the interval $(0,g)$.

Williamson [9] suggests an "averaged" continuity condition of the type

$$\int_0^g F(z)\,dz=0$$

which is independent of z and, contrary to previous formulations [6], involves the continuity condition for each point from $(0,g)$. It should be stressed, however, that such an averaged equation is not equivalent to (6), since satisfying $\int_0^g F(z)\,dz=0$ does not imply that $F(z)$ equals zero for every z belonging to the interval $(0,g)$. Instead, it can easily be shown that (6) is equivalent to an infinite set of equations

$$\int_0^g F(z)\psi_j(z)\,dz=0, \qquad j=0,1,2,\cdots \tag{8}$$

where the functions ψ_j form a complete orthonormal set in the interval $(0,g)$. Indeed, $F(z)$ will vanish if all its Fourier coefficients (given in fact by (8)) vanish simultaneously.

Thus substituting (7) into (8), we find that the resonance condition (5) is equivalent to the infinite set

$$(R^{\mathrm{I}}E_z,\psi_j)+(R^{\mathrm{II}}E_z,\psi_j)=0, \qquad j=0,1,2,\cdots \tag{9}$$

where (u,v) denotes a scalar product $\int_0^g u(z)v(z)\,dz$.

It should be noted that, contrary to other methods, we do not require any information about the electric field distributions on the boundary surface $r=a$. We assume that only $E_z(a,z)$ is at least a piecewise continuous function of z and may be expanded in a series

$$E_z(a,z)=\sum_{i=0}^{\infty} c_i\psi_i(z). \tag{10}$$

On substituting (10) into (9) we obtain an infinite set of homogeneous linear equations

$$\sum_{i=0}^{\infty} c_i\left[(R^{\mathrm{I}}\psi_i,\psi_j)+(R^{\mathrm{II}}\psi_i,\psi_j)\right]=0, \qquad j=0,1,2,\cdots. \tag{11}$$

A nonzero solution for c_i exists only if the determinant formed from the expressions in square brackets vanishes. Hence, the final transcendental equation may be written as

$$\det[W]=0 \tag{12}$$

where elements of the matrix $[W]$ are given by

$$w_{ij}=w_{ji}=(R^{\mathrm{I}}\psi_i,\psi_j)+(R^{\mathrm{II}}\psi_i,\psi_j), \qquad i,j=0,1,2,\cdots. \tag{13}$$

Since the set $\{\psi_i\}$ is infinite, the exact solution of (12) with respect to the resonant frequency could be found for an infinite dimensional matrix only. In practice, however, (12) may be solved approximately to any desired order by truncating both the set of equations (9) and the series (10) at $i,j=n$. Thus extending successively the matrix dimension we obtain a sequence of approximations tending to the exact solution when n tends toward infinity.

It is interesting that the set of linear equations identical with (11) may be derived from Weinstein's variational method of intermediate problems [11], [12]. General variational formulation implies that the sequence of successive approximations is nondecreasing

$$f^{(0)}\leqslant f^{(1)}\leqslant\cdots\leqslant f^{(n)}\leqslant\cdots\leqslant f \tag{14}$$

where f is the exact resonant frequency, and $f^{(n)}$ is the approximate solution of (12) for $i,j\leqslant n$. In other words, approximate solutions $f^{(n)}$ form a monotonic sequence converging to the exact solution from below.

Note that the number of roots of (12) is infinite. The lowest one corresponds to the fundamental mode, whereas the others correspond to higher TM modes of axial symmetry.

III. Numerical Results

Let us take into consideration the following set:

$$\psi_i(z)=\begin{cases}1/\sqrt{g}, & i=0\\ \sqrt{2/g}\,\cos(i\pi z/g), & i=1,2,3,\cdots\end{cases} \tag{15}$$

which is complete and orthonormal in the interval $(0,g)$. After substituting (15) into (13) we obtain diagonal elements

$$w_{ii}=\begin{cases}G_0^{\mathrm{II}}-\dfrac{g}{h}\left\{G_0^{\mathrm{I}}+2\displaystyle\sum_{m=1}^{\infty}\left[\frac{\sin(m\pi g/h)}{m\pi g/h}\right]^2 G_m^{\mathrm{I}}\right\}, & i=0\\ G_i^{\mathrm{II}}-4\dfrac{g}{h}\displaystyle\sum_{m=1}^{\infty}\left[\frac{\sin(m\pi g/h)}{m\pi g/h}\right]^2\frac{G_m^{\mathrm{I}}}{\left[1-(ih/mg)^2\right]^2}, & i=1,2,3,\cdots\end{cases} \tag{16}$$

and off-diagonal elements

$$w_{ij}=\left.\begin{matrix}2^{3/2}\\ 4\end{matrix}\right\}\times(-1)^{m+n+1}\frac{g}{h}\sum_{m=1}^{\infty}\left[\frac{\sin(m\pi g/h)}{m\pi g/h}\right]^2 \cdot\frac{G_m^{\mathrm{I}}}{\left[1-(ih/mg)^2\right]\left[1-(jh/mg)^2\right]}, \quad \begin{matrix}i=0\\ i=1,2,3,\cdots\end{matrix} \tag{17}$$

where G_m^{I} and G_n^{II} are defined in (2a) and (2b).

In order to illustrate the effectiveness of the method, numerical calculations have been performed for a few reentrant cavities investigated earlier by Uenakada [4] and Williamson [9].

TABLE I
SEQUENCE OF SUCCESSIVE APPROXIMATIONS FOR THE RESONANT FREQUENCY
$h=22.792$ mm, $g=7.958$ mm, $a=6.004$ mm, AND $b=42.29$ mm

n	$f^{(n)}$, GHz
0	2.09649
1	2.11819
2	2.12257
3	2.12427
4	2.12514
5	2.12566
6	2.12600
7	2.12624
8	2.12641
9	2.12654
10	2.12665

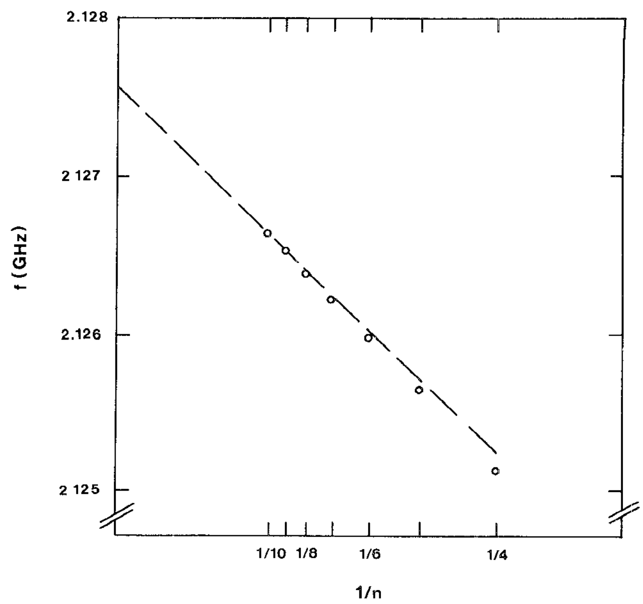

Fig. 2. Extrapolation procedure for the sequence of successive approximations.

First, let us consider resonator 1, whose dimensions are typical for narrow-gap reentrant cavities. Successive approximations obtained from (12) by extending the matrix $[W]$ dimension are listed in Table I.

Note that the experimental result of Uenakada [4] is $f_m=2.135$ GHz, while the theory of Williamson [9] yields $f_w=2.1244$ GHz. It is evident that after a few steps our iterative procedure provides better approximation than Williamson's method: the most accurate among those published so far. Moreover, it turns out that for a sufficiently large matrix dimension successive solutions $f^{(n)}$ are approximately linearly dependent on $1/n$

$$f-f^{(n)}\propto 1/n. \tag{18}$$

Such a dependence is demonstrated in Fig. 2, where numerical results from Table I have been plotted against $1/n$.

It may easily be checked that for $n\geqslant 5$ the deviation from the linear dependence does not exceed ± 0.0001 GHz. Thus on extrapolating the plot to the vertical axis $(n\rightarrow\infty)$ we obtain the resonant frequency $f^{(\infty)}=2.1276$ GHz, which is expected to correspond to the exact solution with uncertainty ± 0.0001 GHz. In other words, owing to the extrapolation procedure, it is possible to reach an accuracy of about 5×10^{-5}, being at least two orders of magnitude better than that attainable by the methods previously published.

Table II presents comparison of experimental data and numerical results obtained by various methods for seven reentrant cavities investigated by Uenakada [4].

Note that the only approximation involved in $f^{(\infty)}$ is due to the extrapolation procedure for the sequence of successive solutions $f^{(n)}$. Therefore, $f^{(\infty)}$ is expected to be very accurate; consequently, it seems justifiable to take $f^{(\infty)}$ as a reference value for comparison with other results.

We can see that both f_w and $f^{(10)}$ are always smaller than $f^{(\infty)}$. This feature is in agreement with variational formulation [12], which implies that any approximation of the aperature field $E_z(a,z)$ provides underestimation of unknown resonant frequency. Error of f_w with respect to $f^{(\infty)}$ ranging from 0.15 to 1.2 percent depends strongly on the relation between cavity dimensions, and reflects to some extent the validity of Williamson's approximation. On the contrary, the accuracy of our results is expected to be independent of the cavity geometry. Thus the method presented here, though very useful for narrow-gap cavities, appears most advantageous when Williamson's formulation becomes inaccurate, e.g., for wider gaps (cavities 5 and 7) or in those cases when the outer cavity wall may influence the aperture field distribution (cavities 2 and 6).

As far as experimental results are concerned, the error of Uenakada's measurements can be estimated to fall within ± 0.4 percent, except for cavity 6 for which f_m is accurate to about 0.8 percent.

IV. CONCLUSIONS

The most relevant features of the method outlined in this paper are summarized below.

1) Contrary to various methods published previously, the method suggested here is valid for arbitrary relations between cavity dimensions.

2) Knowledge of the field distribution on the interface $r=a$ is not required. Conversely, the tangential component of the electric field on the interface may be determined by solving the set (11) for c_i and then substituting to (10).

3) The method is rigorous, i.e., the sequence of successive approximations approaches the exact solution when n

TABLE II
COMPARISON OF EXPERIMENTAL DATA AND THEORETICAL RESULTS
NOTE: f_m = EXPERIMENTAL RESULT OF UENAKADA [4], f_w = THEORETICAL RESULT OF WILLIAMSON [9], $f^{(10)}$ = SUGGESTED METHOD FOR $n=10$, AND $f^{(\infty)}$ = EXTRAPOLATION FOR $n \to \infty$.

Cavity no.	h mm	g mm	a mm	b mm	f_m GHz	f_w GHz	$f^{(10)}$ GHz	$f^{(\infty)}$ GHz
1	22.792	7.958	6.004	42.29	2.135	2.1244	2.1267	2.1276
2	34.826	8.028	5.992	13.80	2.326	2.3086	2.3297	2.3310
3	31.806	7.984	5.9935	20.99	2.280	2.2678	2.2782	2.2796
4	28.019	7.999	5.999	29.988	2.2264	2.2164	2.2228	2.2241
5	31.806	7.980	3.495	20.99	2.394	2.3749	2.3952	2.3970
6	33.806	10.000	8.405	20.99	2.3027	2.2689	2.2844	2.2859
7	33.806	10.000	4.206	20.99	2.4018	2.3789	2.4082	2.4104

tends toward infinity. The only approximation which is made in practice consists in truncating the sum (10) and the set (11) at the nth term. In other words, required accuracy can always be achieved if a sufficiently large matrix $[W]$ is taken into consideration.

4) The sequence of successive approximations resulting from successive expansion of matrix dimensions is nondecreasing. Thus for any finite n, an approximate solution underestimates real resonant frequency.

5) The method is very effective; the 10×10 matrix provides much better approximation than any method published so far. Moreover, the sequence of successive approximations, when plotted against $1/n$, can be easily extrapolated. Such a procedure results in an estimation with the error of about 5×10^{-5} for typical reentrant cavities.

6) Not only a fundamental mode but also any higher TM mode of axial symmetry may be found by choosing an appropriate solution of (12).

7) Numerical problems are slightly more complex than those arising from Williamson's approach. Fortunately, reasonable accuracy may be obtained for low-order matrices ($n<10$), so numerical procedure is not very time consuming.

Finally, it should be noted that applicability of the method presented here is not confined to reentrant cylindrical cavities only. The general idea appears suitable for any resonant system which can be divided into sufficiently regular subregions.

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